Power Radiated by a Point Charge

The fields of a point charge q in arbitrary motion

$$E(r,t) = \frac{q}{4\pi\varepsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot u)^3}\left[(c^2-v^2)u + \vec{\mathscr{r}}\times(u\times a)\right], \quad \text{---(1)}$$

where $u = c\hat{\mathscr{r}} - v$

and $B(r,t) = \frac{1}{c}\hat{\mathscr{r}}\times E(r,t)$ ---(2)

First term of eq^n (1) → velocity field
Second term → acceleration field

The Poynting vector is

$$S = \frac{1}{\mu_0}(E\times B)$$

$$= \frac{1}{\mu_0 c}\left[E\times(\hat{\mathscr{r}}\times E)\right]$$

$$= \frac{1}{\mu_0 c}\left[E^2\hat{\mathscr{r}} - (\hat{\mathscr{r}}\cdot E)E\right] \quad \text{---(3)}$$

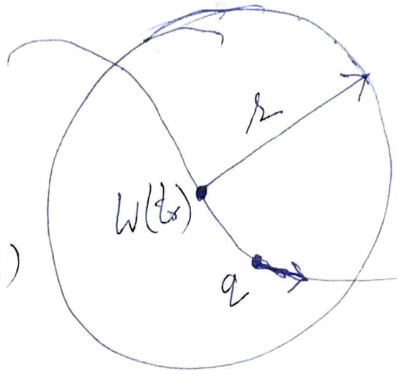
Not all of this energy flux constitutes → radiation; some of it is just field energy carried along by the particles as it moves.

Radiated energy → detaches itself from the charge and propagates off to infinity.

To calculate the total power radiated by the particle at time t_r, we draw a huge sphere of radius $\mathscr{r}$, centered at the position of the particle (at time t_r)

Wait the appropriate interval

$t - t_r = \frac{r}{c}$ — (4)



for the radiation to reach the sphere, and at that moment integrate the Poynting vector over the surface.

Area of the sphere $\propto r^2$, only terms that goes like $\frac{1}{r^2}$ will yield a finite answer but terms like $\frac{1}{r^3}$ & $\frac{1}{r^4}$ will contribute nothing in the limit $r \to \infty$.

$\Rightarrow$ Only the acceleration fields represent true radiation
↳ radiation fields

$$E_{rad} = \frac{q}{4\pi\epsilon_0} \frac{r}{(\vec{r}\cdot u)^3}\left[\vec{r} \times (u \times a)\right]$$ — (5)

Velocity fields → carry energy and as charge moves this energy is dragged along → but it is not radiation

Now E_{rad} is perpendicular to $\hat{r}$

So second term in eq^n (3) vanishes

$$S_{rad} = \frac{1}{\mu_0 c} E_{rad}^2 \hat{r}$$ — (6)

If the charge is instantaneously at rest (at time t_r), then $u = c\hat{r}$ and

$$E_{rad} = \frac{q}{4\pi\epsilon_0 c^2 r}\left[\hat{r} \times (\hat{r} \times a)\right]$$

$$= \frac{\mu_0 q}{4\pi r}\left[(\hat{r}\cdot a)\hat{r} - a\right] \quad --(7)$$

In that case

$$S_{rad} = \frac{1}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi r}\right)^2\left[a^2 - (\hat{r}\cdot a)^2\right]\hat{r}$$

$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\hat{r} \quad --(8)$$

$\theta \rightarrow$ angle between $\hat{r}$ and a.

No power is radiated in the forward or backward direction — rather, it is emitted in a donut about the direction of instantaneous acceleration

The total power radiated is

$$P = \oint S_{rad}\cdot da$$

$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c}\int \frac{\sin^2\theta}{r^2} r^2 \sin\theta \, d\theta \, d\phi$$

$$\boxed{P = \frac{\mu_0 q^2 a^2}{6\pi c}} \quad --(9)$$

$\hookrightarrow$ Larmor formula

We derived it on the assumption that $v=0$
eq^n's (8) & (9) actually hold to good approximation
as long as $v \ll c$

Liénard generalization of the Larmor formula

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{v \times a}{c}\right|^2\right) \quad \text{--- (10)}$$

where $\gamma \equiv 1/\sqrt{1-\frac{v^2}{c^2}}$

The factor γ^6 means that the radiated power increases enormously as the particle velocity approaches the speed of light.

Bremsstrahlung:—

Accelerated charges → radiate

Acceleration may be positive or negative

deceleration also produce radiation?

Beam of electrons is projected into a block of material → electrons ~~are~~ are stopped and radiation is produced

↳ This radiation is called Bremsstrahlung (i.e. "braking radiation")

When the velocity & acceleration are perpendicular. → Circular motion

→ radiation is called synchrotron radiation.

Radiation Reaction

Laws of classical electrodynamics →

an accelerating charge radiates

This radiation carries off energy → at the expense of the particle's kinetic energy.

Under the influence of a given force → a charged particle accelerates less than a neutral one of the same mass.

The radiation evidently exerts a force (F_{rad}) back on the charge → a recoil force like that of a bullet on a gun

We derive the radiation reaction force from conservation of energy

For a non-relativistic particle ($v \ll c$) the total power radiated is given by the Larmor formula

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \quad \text{---(1)} \quad eq^n (9)$$

Conservation of energy suggests that this is also the rate at which the particle loses energy.

under the influence of the radiation reaction force F_{rad}:

$$F_{rad} \cdot v = -\frac{\mu_0 q^2 a^2}{6\pi c} \quad \text{--(2)}$$

As the particle accelerates and decelerates energy is exchanged between it and the velocity fields, and at the same time as energy is irretrievably radiated away by the acceleration fields.

Eqn (2) only accounts for the latter, but if we want to know the recoil force exerted by the fields on the charge, we need to consider the total power loss at any instant, not just the portion that eventually escapes in the form of radiation.

We should really call it "field reaction" not radiation reaction.

Eqn (2), while incorrect instantaneously, is valid on the average:

$$\int_{t_1}^{t_2} F_{rad} \cdot v \, dt = -\frac{\mu_0 q^2}{6\pi c} \int_{t_1}^{t_2} a^2 \, dt \quad \text{--(3)}$$

with the stipulation that the "state of the system" is identical at t_1 and t_2.

R.H.S can be integrated by parts

$$\int_{t_1}^{t_2} a^2 dt = \int_{t_1}^{t_2} \left(\frac{dv}{dt}\right)\left(\frac{dv}{dt}\right) dt = \left(v \cdot \frac{dv}{dt}\right)\Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d^2 v}{dt^2} \cdot v \, dt$$

The boundary term drops out, since the velocities and accelerations are identical at t_1 and t_2 so eqn (3) can be written

$$\int_{t_1}^{t_2} \left(F_{rad} - \frac{\mu_0 q^2}{6\pi c} \dot{a}\right) \cdot v \, dt = 0 \quad \text{--(4)}$$

Eq^n (4) will certainly be satisfied if

$$\boxed{F_{rad} = \frac{\mu_0 q^2}{6\pi c}\dot{a}} \quad \text{--- (5)}$$

This is the Abraham–Lorentz formula for the radiation reaction force. Simplest form of the radiation reaction force, consistent with conservation of energy.

Prob: Calculate the radiation damping of a charged particle attached to a spring of natural frequency ω_0, driven at frequency ω.

Sol^n The eq^n of motion is

$$m\ddot{x} = F_{spring} + F_{rad} + F_{driving}$$

$$= -m\omega_0^2 x + m\tau\dddot{x} + F_{driving}$$

$$\tau = \frac{\mu_0 q^2}{6\pi m c}$$

With the system oscillating at frequency ω

$$x(t) = x_0\cos(\omega t + \delta)$$

so

$$\dddot{x} = -\omega^2\dot{x}$$

Therefore $m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = F_{driving}$ --- (6)

and the damping factor γ is given by

$$\boxed{\gamma = \omega^2\tau} \quad \text{--- (7)}$$

Radiation damping is, at least, is proportional to $\ddot{v}$